Estimating Parameter Heterogeneity in Decision Programs by Spatial Quasi-Differentiation (SQD)

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Extended Abstract

Motivation. Many systems across the social sciences are governed by decision programs whose structural parameters vary with characteristics of agents or locations. In discrete choice models, for example, unobserved heterogeneity in consumer valuations is central to understanding demand. Standard approaches to recovering such heterogeneity typically require the researcher to impose parametric distributional assumptions on the heterogeneous parameters, for instance specifying that random coefficients follow a particular distributional family. Nonparametric alternatives exist, but often carry substantial computational costs or require specialized estimators tailored to the specific model. A key question is therefore: *can we recover the distribution of heterogeneous parameters in a model-free way, without imposing distributional structure, and in a computationally tractable manner?*

We address this question by extending the method of quasi-differentiation (QD), a method introduced for analysing noisy time series from complex systems using sliding pairs of adjoining windows [1], to estimation of parameter heterogeneity in a cross-sectional, or spatial, domain. The key insight is that if heterogeneous parameters vary smoothly with observable covariates, local differencing of estimator outputs across the covariate space recovers the derivative of the parameter mapping—without specifying what that mapping is. Reintegrating these quasi-derivatives then recovers the heterogeneous parameter profile in a fully model-free fashion.

Approach and Methodology. A decision program $\mathfrak{P}$ is characterized by the parameter vector θ. We have a sample $\mathbf{S}_N$ of N observations in $\mathbb{R}^D$ from this program, i.e., $\mathbf{S}_N = \{s_n\}_{n=1}^N \in (\mathbb{R}^D)^N$. Our interest is in the estimator $\sigma(\mathbf{S}_N;\mathfrak{P}) : (\mathbb{R}^D)^N \mapsto \mathbb{R}^{|\theta|}$, where θ is the parameter vector of interest. In particular, we consider the case where θ is heterogeneous across the sample, i.e., $\theta = \{\theta_n\}_{n=1}^N$. The heterogeneity in θ_n is driven by observable covariates, but the functional form of the mapping from covariates to parameters is unknown.

SQD proceeds in three steps. In the first step, for the case of a single covariate x_n, let $\mathbf{S}_N$ be ordered by x_n so that $n' > n \Rightarrow x_{n'} > x_n$. In the second step, for each observation n, two adjacent local subsamples are formed: a right window containing observations $\{x' \in x(\mathbf{S}_N) \mid x_n \leq x' \leq x_n + w\}$ and a left window $\{x' \in x(\mathbf{S}_N) \mid x_n \geq x' \geq x_n - w\}$, where w is the window width. The estimator σ is applied separately to each subsample, and the difference defines the quasi-differentiation operator:

$$\Delta^w \sigma(s_n, \mathbf{S}_N; \mathfrak{P}) \to \widehat{\theta}^w(s_n). \tag{1}$$

In the third step, the collection of quasi-derivatives across observations is reintegrated:

$$\left\{\widehat{\theta}^w(s_n)\right\}_{n=1}^N \to \left\{\widehat{\theta}(s_n)\right\}_{n=1}^N, \tag{2}$$

which is accomplished by reintegration and subsequent stochastic inversion of the parameter levels. The window width w governs a bias-variance tradeoff: narrow windows produce quasi-derivatives with low bias but high variance, while wide windows smooth the estimates at the cost of attenuating sharp features in the parameter profile.

A key property of the method is that we can recover the heterogeneity in parameters without explicitly introducing this heterogeneity in the program nor in the initial estimation stage, which allows a model-free approach for heterogeneity—where we typically have less theoretical justification for a particular structure—and simplifies the computational complexity.

Results. We are developing several applications of SQD to demonstrate its ability to recover parameter heterogeneity across different decision programs, with results to be completed by the time of the conference.

The first application is a demand estimation problem with a binary logit discrete choice model. In each of N markets, M consumers choose whether to purchase a product, with market-specific valuations v_n that depend on an observable covariate x_n through an unknown mapping. For a data set $\mathbf{S}$, $\sigma(\mathbf{S}; \mathfrak{P})$ is the maximum likelihood estimator of $\{v\}$, applied to local subsamples via SQD. Reintegration and anchoring recover the level of the heterogeneous valuation profile by matching the empirical mean quantity. Monte Carlo simulations across multiple window widths and replications will evaluate the accuracy of the recovered profiles relative to the true data-generating process.

A second application extends SQD to a monopoly pricing problem, where market-specific cost parameters enter through an optimality condition rather than directly through a likelihood. A further extension to multi-dimensional covariate spaces is also under development.

Conclusions and Outlook. The proposed method offers a general, model-free approach to recovering parameter heterogeneity in decision programs by differencing estimator outputs across local subsamples in the covariate space and reintegrating. The method is applicable wherever a point estimator is available and heterogeneity varies smoothly with observables, bridging complex systems methodology and structural estimation across disciplines.

References

[1] Siew Ann Cheong, Zheng Tien Kang, and Peter Tsung-Wen Yen. Quasi-differentiation and its applications to noisy time series data from complex systems. *Scientific Reports*, 15:39080, 2025.