Dimensions of hypergraphs using spreading processes

Keywords: spreading dynamics, fractals, hypergraphs, higher-order interactions, complex networks

Extended Abstract

Motivation. Fractal dimension is a generalisation of the topological dimension and measures how particular structural properties of an object scale with across different spatial sizes. Unlike the Euclidean case, fractal dimensions are not constrained to integer values and may instead take on any real value. For continuous subsets $X \subseteq \mathbb{R}^d$, the fractal dimension can be calculated using a variety of methods such as the box-counting and correlation dimension. Commonly, one tracks a quantity $C(s)$ representing a notion of volume or bulk of X measured at a spatial resolution s. The fractal dimension is defined as the exponent d of power law given by

$$C(s) \sim s^d.$$

For simple geometric objects, the fractal dimension is equal to the Euclidean dimension (lines $d = 1$, surfaces $d = 2$, volumes $d = 3$). However, this property is not preserved when one considers more irregular sets.

Non-integer values of fractal dimension are commonly encountered in attractors of dynamical systems where non-integer values of fractal dimension are often interpreted as indicators of chaotic behaviour. Because their dynamical invariance, fractal dimensions can be used to compare the quality of reconstructions for chaotic attractors. Here, fractal dimensions is often taken to be equal to the correlation dimension where there exists several well established numerical estimation methods such as those by Grassberger & Procaccia[1].

Notions of fractal dimension have also been proposed for networks to quantify similar features of self-similarity and spatial scaling. Values of fractal dimension allow for the inclusion of irregular structures of networks to be included in a given mode. The network box counting dimension [5] has been used to calculate the vulnerability of transport networks [6], and as a size agnostic measure of network robustness [7]. More recently, network correlation dimension was used to construct continuous approximations of network disease spread that accurately describe early stages of an outbreak [4].

Recently, extensions of disease dynamics on graphs have been extended to the case of hypergraphs presented as simplicial contagion models[2]. The inclusion of dynamics on higher-order simplices have been shown to produce non-trivial behaviours such as reduced epidemic thresholds and bistability with validations against mean-field continuous approximations[3]. However, the latter methods become unwieldy when considering hyperedges or higher order. The fractal dimension of a hypergraph is of particular interest as it provides a potential method for including the scaling structure into continuous approximations.

Approach and Methodology. The estimation of fractal dimensions fundamentally relies on defining notions of distance between elements in the set of interest. In the case of point cloud data from dynamical system trajectories, the standard Euclidean distance is used to define the threshold sizes for calculating the box-counting and correlation dimensions. In their corresponding network extensions, the shortest path length (i.e. hop count) is the natural choice. However, these methods are not directly applicable when considering the structure of hypergraphs.

Hypergraphs – a generalisation of networks – allow for the inclusion of hyperedges that represent connections between a group of nodes of arbitrary size. This improved flexibility also poses a challenge for defining the distances required to calculate fractal dimensions. Specifically, there is ambiguity on how the contribution to the shortest distance varies between hyperedges of different order. This talk will discuss the challenges associated with defining and enumerating distances on hypergraphs, and present a method to estimate notions of fractal dimension for general hypergraphs using spreading processes as proxies for distance.

Results. We demonstrate how spreading processes are able to replicate the network dimensions of particular regular graphs and show existence of scaling law relationships. We also present preliminary results of the method applied to various empirical networks and hypergraphs.

Conclusions and Outlook. Further refinements and validation for the method will be done, and subsequently applied to a wider set of empirical networks.

References

[1] Peter Grassberger and Itamar Procaccia. Measuring the strangeness of strange attractors. *Physica D*, 9(1-2):189–208, 1983.

[2] Iacopo Iacopini, Giovanni Petri, Alain Barrat, and Vito Latora. Simplicial models of social contagion. *Nature Communications*, 10(1):2485, 2019.

[3] Federico Malizia, Alessandra Corso, Lucia Valentina Gambuzza, Giovanni Russo, Vito Latora, and Mattia Frasca. Reconstructing higher-order interactions in coupled dynamical systems. *Nature Communications*, 15(1):5184, 2024.

[4] Jack Murdoch Moore, Michael Small, Gang Yan, Huijie Yang, Changgui Gu, and Haiying Wang. Network spreading from network dimension. *Physical Review Letters*, 132(23):237401, 2024.

[5] Chaoming Song, Lazaros K Gallos, Shlomo Havlin, and Hernán A Makse. How to calculate the fractal dimension of a complex network: the box covering algorithm. *Journal of Statistical Mechanics*, 2007(03):P03006–P03006, 2007.

[6] Tao Wen, Moxian Song, and Wen Jiang. Evaluating topological vulnerability based on fuzzy fractal dimension. *International Journal of Fuzzy Systems*, 20(6):1956–1967, 2018.

[7] Yipeng Wu, Zhilong Chen, Kui Yao, Xudong Zhao, and Yicun Chen. On the correlation between fractal dimension and robustness of complex networks. *Fractals*, 27(04):1950067, 2019.