Steady-state Response Theory in Noisy Network Dynamics

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Abstract

Many complex interacting systems in physics, biology, and engineering can be modeled as network dynamics under the influence of stochastic noises. Examples include neural networks, ecological interactions, and epidemic networks. Depending on the interplay between network topology and local dynamics, these stochastic network systems often converge to a stable steady state. Noise sources can be treated as environmental factors or stochastic perturbations, which lead to fluctuations around steady states. To understand the influence of perturbations on complex systems, we applied the linear response theory (LRT), which is a powerful tool in statistical mechanics to predict the systems' response. While linear response theory (LRT) is well-established in equilibrium statistical physics, its extension to nonequilibrium steady states (NESS) on complex networks is much less explained. In particular, it is unclear how network topology and heterogeneous noise jointly shape the steady-state response. In this work, we develop a steady-state response theoretical framework for stochastic network dynamics, focusing on overdamped systems near stable NESS.

We study overdamped Langevin dynamics on networks, where nodes $x_i(t)$ evolve under deterministic intrinsic (nonlinear) dynamics and linearized coupling interactions, and each node is driven by Gaussian white noise. Asymmetric couplings and heterogeneous noise break detailed balance, leading to NESS. The general noisy network dynamics of N nodes is governed by

$$\dot{x}_i = f_i(x_i; r_i) + \sum_{j\neq i} W_{ij} h(x_j, x_i),\ i = 1, 2, ..., N, \tag{1}$$

where r_i is the intrinsic node parameter and W_{ij} is the connection weight from node j to i, and h is some coupling function between node i and j. To verify our theoretical results, we generate an ER random network with N nodes and connection probability p. The node

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parameters and connection weights are drawn from a prescribed distribution. The validity of the linear response theory is verified by comparing the simulation (or analytic) results with the analytic derived expression of the response function. We further extend the framework to time-dependent perturbations, expressing the non-steady response as a convolution between the perturbation rate and the response function.

The response function of various observable can be derived, including the covariance matrix $\mathbf{K}_0 \equiv \langle \delta\vec{x}\delta\vec{x}^{\intercal} \rangle$. It is demonstrated that the response is determined by the unperturbed NESS statistics. Furthermore, for simultaneous perturbations of multiple nodes or multiple edges, the total steady-state response satisfies the linear superposition principle. Simulation results confirm that even under multiple small perturbations or partial disruptions of edges (e.g., setting selected edge weights to zero), the predicted linear combination accurately describes the global change in the observable. To sum up, these results show that steady-state response in noisy networks is (i) predictable from pre-perturbed correlations at nonequilibrium steady states, (ii) structurally determined by network topology, and (iii) linearly decomposable for multiple simultaneous perturbations.

Our findings establish a generalized steady-state linear response theory for noisy network dynamics, applicable to both equilibrium and nonequilibrium steady states, and we derive explicit results for steady or non-steady (time-dependent) response functions subject to parameter(s) changes. Beyond linear regime, we have completed the analytical derivation of second-order (nonlinear) response corrections. Numerical simulations confirm that the second-order response significantly improves prediction accuracy compared to purely linear response.

Figure 1: Verification of steady-state linear response in noisy networks. (a) Covariance response $d\mathbf{K}_0/dr_\alpha$ versus theoretical response function $\chi_{ij\alpha}$ under perturbations of intrinsic parameters. (b) Covariance response $d\mathbf{K}_0/dW_{\alpha\beta}$ versus theoretical response function $\chi_{ij\alpha\beta}$ under edge weight perturbations.